5. Introduction to Robot Geometry and Kinematics

The goal of this chapter is to introduce the basic terminology and notation used in robot geometry and kinematics, and to discuss the methods used for the analysis and control of robot manipulators. The scope of this discussion will be limited, for the most part, to robots with planar geometry. The analysis of manipulators with three-dimensional geometry can be found in any robotics text[1].

5.1 Some definitions and examples

We will use the term mechanical system to describe a system or a collection of rigid or flexible bodies that may be connected together by joints. A *mechanism* is a mechanical system that has the main purpose of transferring motion and/or forces from one or more sources to one or more outputs. A *linkage* is a mechanical system consisting of rigid bodies called *links* that are connected by either pin joints or sliding joints. In this section, we will consider mechanical systems consisting of rigid bodies, but we will also consider other types of joints.

Degrees of freedom of a system

> *The number of independent variables (or coordinates) required to completely specify the configuration of the mechanical system.*

While the above definition of the number of degrees of freedom is motivated by the need to describe or analyze a mechanical system, it also is very important for controlling or driving a mechanical system. It is also the number of independent inputs required to drive all the rigid bodies in the mechanical system.

Examples:

(a) A point on a plane has two degrees of freedom. A point in space has three degrees of freedom.

(b) A pendulum restricted to swing in a plane has one degree of freedom.

[1]In particular, two books offer an excellent treatment while keeping the mathematics at a very simple level: (a) Craig, J. J. *Introduction to Robotics*, Addison-Wesley, 1989; and (b) Paul, R., *Robot Manipulators, Mathematics, Programming and Control*, The MIT Press, Cambridge, 1981.

(c) A planar rigid body (or a lamina) has three degrees of freedom. There are two if you consider translations and an additional one when you include rotations.

(d) The mechanical system consisting of two planar rigid bodies connected by a pin joint has four degrees of freedom. Specifying the position and orientation of the first rigid body requires three variables. Since the second one rotates relative to the first one, we need an additional variable to describe its motion. Thus, the total number of independent variables or the number of degrees of freedom is four.

(e) A rigid body in three dimensions has six degrees of freedom. There are three translatory degrees of freedom. In addition, there are three different ways you can rotate a rigid body. For example, consider rotations about the x, y, and z axes. It turns out that any rigid body rotation can be accomplished by successive rotations about the x, y, and z axes. If the three angles of rotation are considered to be the variables that describe the rotation of the rigid body, it is evident there are three rotational degrees of freedom.

(f) Two rigid bodies in three dimensions connected by a pin joint have seven degrees of freedom. Specifying the position and orientation of the first rigid body requires six variables. Since the second one rotates relative to the first one, we need an additional variable to describe its motion. Thus, the total number of independent variables or the number of degrees of freedom is seven.

Kinematic chain

A system of rigid bodies connected together by joints. A chain is called closed if it forms a closed loop. A chain that is not closed is called an open chain.

Serial chain

If each link of an open chain except the first and the last link is connected to two other links it is called a serial chain.

An example of a serial chain can be seen in the schematic of the PUMA 560 series robot[2], an industrial robot manufactured by Unimation Inc., shown in Figure 1. The trunk is bolted to a fixed table or the floor. The shoulder rotates about a vertical axis with respect to the trunk. The upper arm rotates about a horizontal axis with respect to the shoulder. This rotation is the shoulder joint rotation. The forearm rotates about a horizontal axis (the elbow) with respect to the upper arm. Finally, the wrist consists of an assembly of three rigid bodies with three

[2]The Programmable Universal Machine for Assembly (PUMA) was developed in 1978 by Unimation Inc. using a set of specifications provided by General Motors.

additional rotations. Thus the robot arm consists of seven rigid bodies (the first one is fixed) and six joints connecting the rigid bodies.

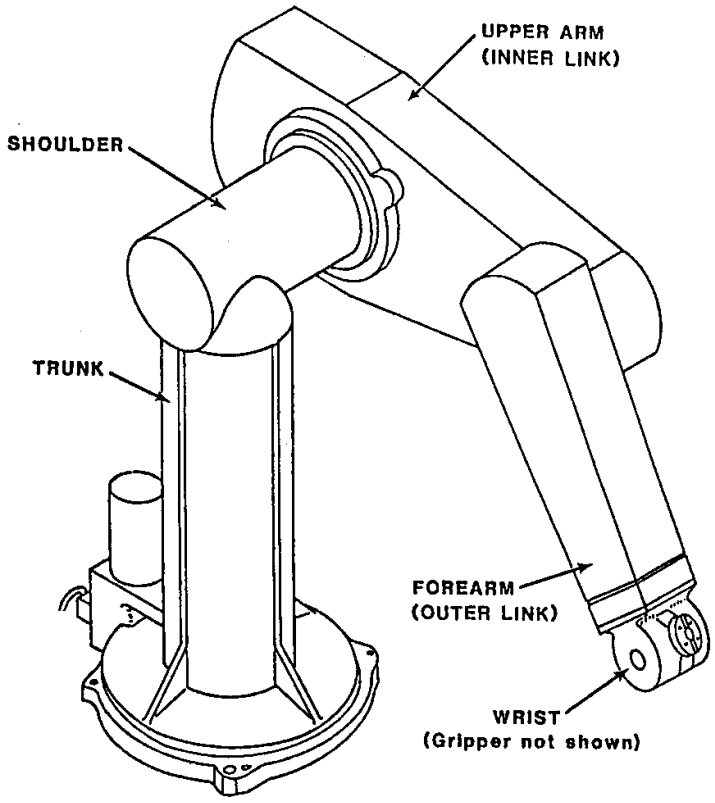


Figure 1 The six degree-of-freedom PUMA 560 robot manipulator.

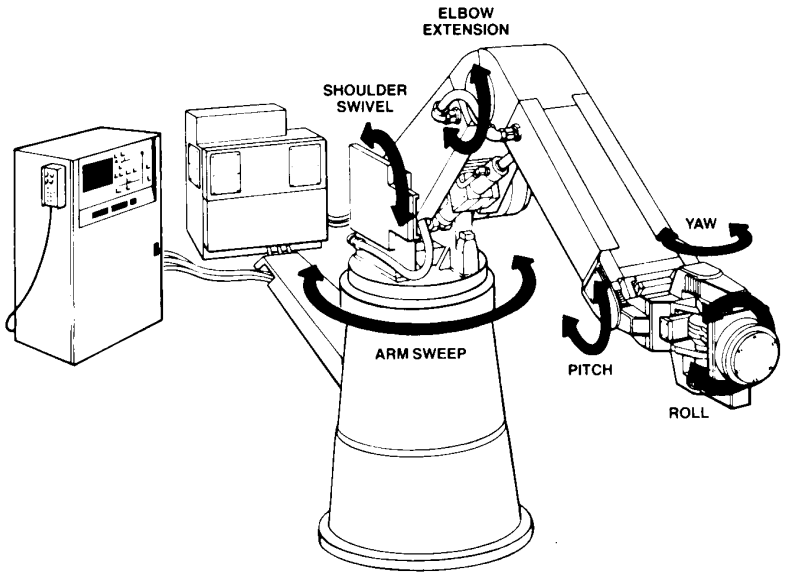


Figure 2 The six degree-of-freedom T3 robot manipulator.

Another schematic of an industrial robot arm, the T3 made by Cincinnati Milacron, is shown in Figure 2. Once again, it is possible to model it as a collection of seven rigid bodies (the first being fixed) connected by six joints[3].

Types of joints

There are mainly four types of joints that are found in robot manipulators:

- Revolute, rotary or pin joint (*R*)
- Prismatic or sliding joint (*P*)
- Spherical or ball joint (*S*)
- Helical or screw joint (*H*)

The revolute joint allows a rotation between the two connecting links. The best example of this is the hinge used to attach a door to the frame. The prismatic joint allows a pure translation between the two connecting links. The connection between a piston and a cylinder in an internal combustion engine or a compressor is via a prismatic joint. The spherical joint between two links allows the first link to rotate in all possible ways with respect to the second. The best example of this is seen in the human body. The shoulder and hip joints, called ball and socket joints, are spherical joints. The helical joint allows a helical motion between the two connecting bodies. A good example of this is the relative motion between a bolt and a nut.

Planar chain

All the links of a planar chain are constrained to move in or parallel to the same plane.

A planar chain can only allow prismatic and revolute joints. In fact, the axes of the revolute joints must be perpendicular to the plane of the chain while the axes of the prismatic joints must be parallel to or lie in the plane of the chain.

An example of a planar chain is shown in Figure 3. Almost all internal combustion engines use a slider crank mechanism. The high pressure of the expanding gases in the combustion chamber is used to translate the piston and the mechanism converts this translatory movement into the rotary movement of the crank. This mechanical system consists of three revolute joints and one prismatic joint.

The example in Figure 3 is a planar, closed, kinematic chain. Examples of planar, serial chains are shown in Figure 4 and 5.

[3]This is a convenient model. A more accurate kinematic model is required to model the coupling between the actuator that drives the elbow joint and the elbow joint.

Connectivity of a joint

The number of degrees of freedom of a rigid body connected to a fixed rigid body through the joint.

The revolute, prismatic and helical joint have a connectivity 1. The spherical joint has a connectivity of 3. Sometimes one uses the term "degree of freedom of a joint" instead of the connectivity of a joint.

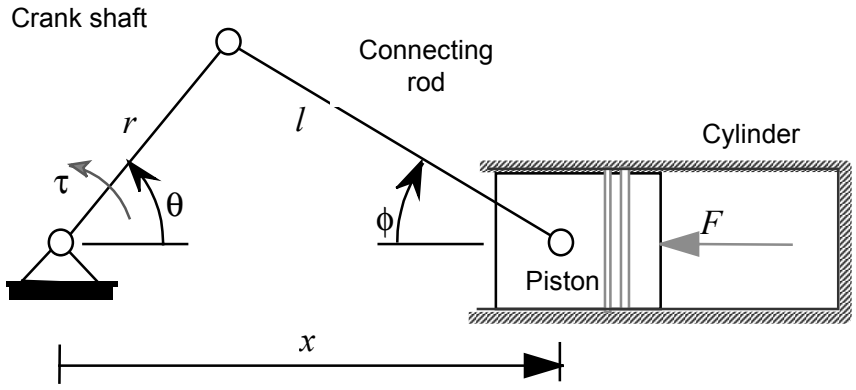


Figure 3 A schematic of a slider crank mechanism

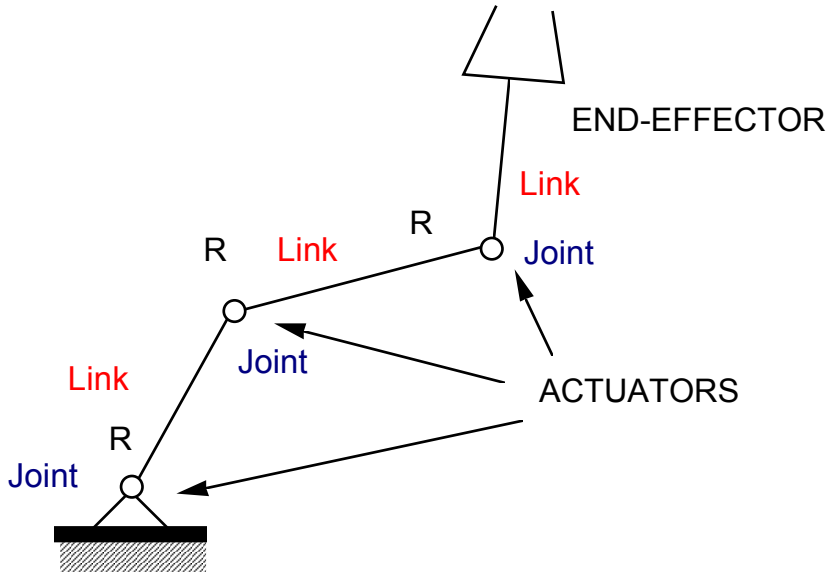


Figure 4 A schematic of a planar manipulator with three revolute joints

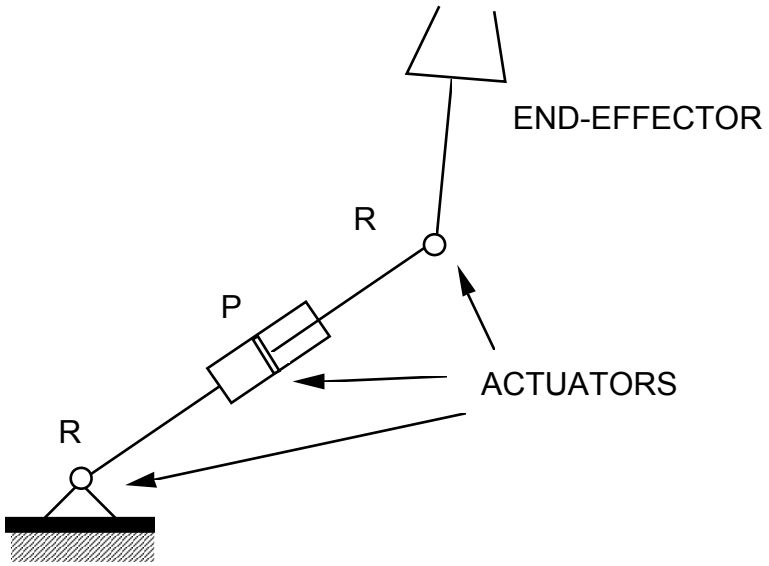


Figure 5 A schematic of a planar manipulator with two revolute and one prismatic joints

Mobility

The mobility of a chain is the number of degrees of freedom of the chain.

Most books will use the term "number of degrees of freedom" for the mobility. In a serial chain, the mobility of the chain is easily calculated. If there are n joints and joint i has a connectivity f_i,

$$M = \sum_{i=1}^{n} f_i$$

Most industrial robots have either revolute or prismatic joints ($f_i = 1$) and therefore the mobility or the number of degrees of freedom of the robot arm is also equal to the number of joints. Sometimes, an n degree of freedom robot or a robot with mobility n is also called an n axis robot.

Since a rigid body in space has six degrees of freedom, the most general robots are designed to have six joints. This way, the end effector or the link that is furthest away from the base can be made to assume any position or orientation (within some range). However, if the end effector needs to moved around in a plane, the robot need only have three degrees of freedom. Two examples[4] of planar, three degree of freedom robots (technically, mobility three robots) are shown in Figures 4 and 5.

[4]Note that we do not count the opening and closing of the gripper as a degree of freedom. The gripper is usually completely open or completely shut and it is not continuously controlled as the other joints are. Also, the gripper freedom does not participate in the positioning and orienting of a part held by the gripper.

When closed loops are present in the kinematic chain (that is, the chain is no longer serial, or even open), it is more difficult to determine the number of degrees of freedom or the mobility of the robot. But there is a simple formula that one can derive for this purpose.

Let n be the number of moving links and let g be the number of joints, with f_i being the connectivity of joint i. Each rigid body has six degrees if we consider spatial motions. If there were no joints, since there are n moving rigid bodies, the system would have $6n$ degrees of freedom. The effect of each joint is to constrain the relative motion of the two connecting bodies. If the joint has a connectivity f_i, it imposes $(6\text{-}f_i)$ constraints on the relative motion. In other words, since there are f_i different ways for one body to move relative to another, there $(6\text{-}f_i)$ different ways in which the body is constrained from moving relative to another. Therefore, the number of degrees of freedom or the mobility of a chain (including the special case of a serial chain) is given by:

$$M = 6n - \sum_{i=1}^{g} (6 - f_i)$$

or,

$$M = 6(n - g) + \sum_{i=1}^{g} f_i \tag{2}$$

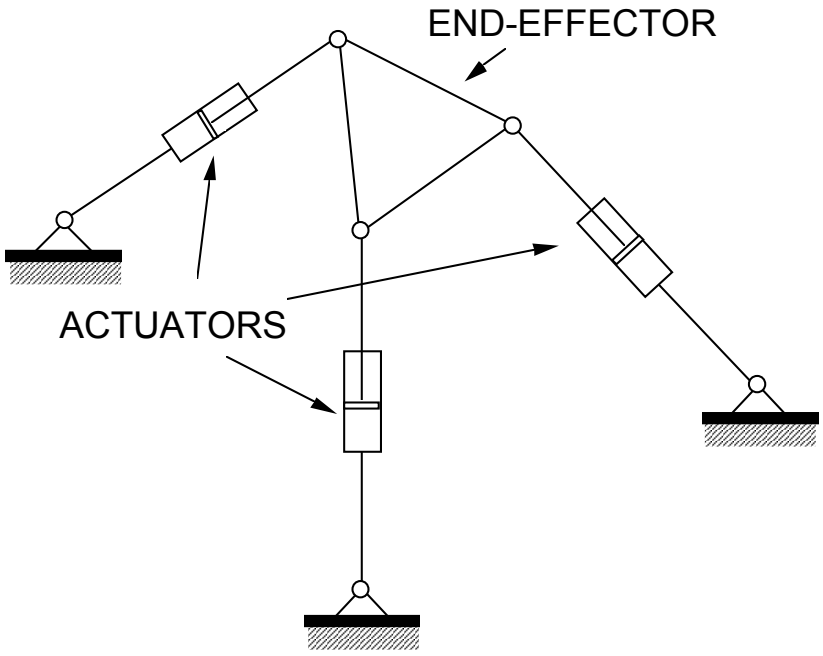


Figure 6 A planar parallel manipulator.

In the special case of planar motion, since each unconstrained rigid body has 3 degrees of freedom, this equation is modified to:

$$M = 3(n-g) + \sum_{i=1}^{g} f_i \tag{3}$$

Example 1

In Figures 4 and Figure 5, since n=g=3, Equation (3) reduces to the special case of Equation (1). And since $f_1 = f_2 = f_3 = 1$, and M=g.

Example 2

In the slider crank mechanism shown in Figure 3, n=3 and$g$=4. Since it is a planar mechanism we use Equation (2). All four joints have connectivity one: $f_1 = f_2 = f_3 = f_4 = 1$, and M=1.

Example 3

Consider the parallel manipulator shown in Figure 6. Here, n = 7, g=9, and f_i=1. According to Equation (3), M =3. There are correspondingly three actuators in the manipulator. Contrast this arrangement with the arrangement shown in Figures 4 and 5. The three actuators are mounted in parallel in Figure 6. In Figures 4 and 5, they are mounted "sequentially" in a serial fashion.

The Stewart Platform

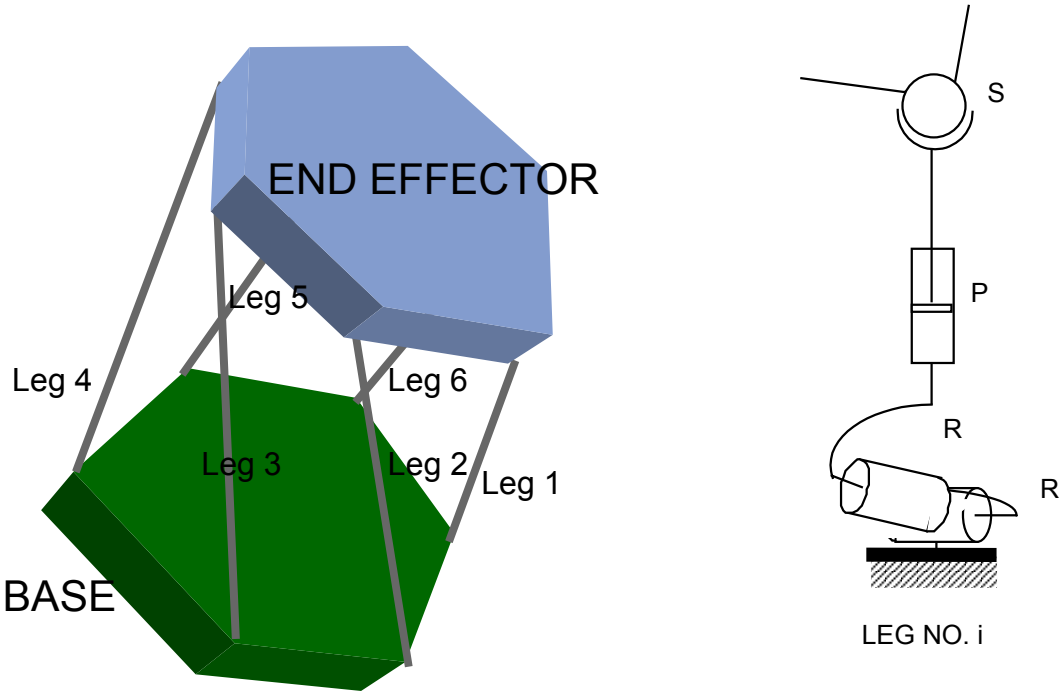
The Stewart-Gough or the Stewart Platform[5] device is a six degree of freedom (mobility six) kinematic chain with closed loops. The kinematic chain consists of a base and a moving platform each of which is a spatial hexagon. See Figure 7. Every vertex of the base hexagon is connected to one vertex of the moving platform hexagon by one leg. Similarly, every vertex of the moving hexagon is connected to a vertex of the base hexagon by a leg. There are six such legs. Each leg has is a serial chain consisting of two revolute joints with intersecting axes, a prismatic joint and a spherical joint. Typically the prismatic joints are actuated.

The mobility of a Stewart Platform can be easily verified to be six. Each leg has three links and four joints. If we include the moving platform,

$n = 6 \times 3+1 = 19$.

[5]D. Stewart, "A Platform with Six Degrees of Freedom," *The Institution of Mechanical Engineers*, Proceedings 1965-66, Vol. 180 Part 1, No. 15, pages 371-386.

(a) A machine tool based on the Stewart Platform (Ingersoll Rand)[6]



(b) A schematic showing the six legs (left) and the *RRPS* chain (right).

Figure 7 The Stewart Platform

[6]M. Valenti, "Machine Tools Get Smarter," *Mechanical Engineering*, Vol.117, No.11, November 1995.

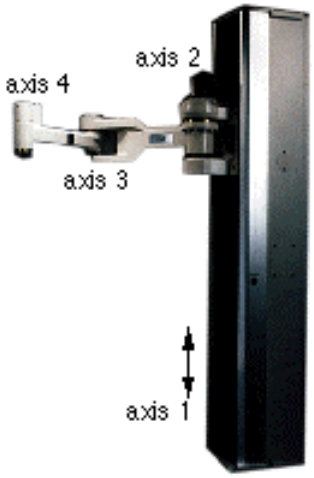
The connectivity of the revolute and the prismatic joint is one. The connectivity of the spherical joint is three. Since there are 6×2 revolute joints, 6 prismatic joints and 6 spherical joints,

$$\sum_{i=1}^{g} f_i = 12 + 6 + 6 \times 3 = 36$$

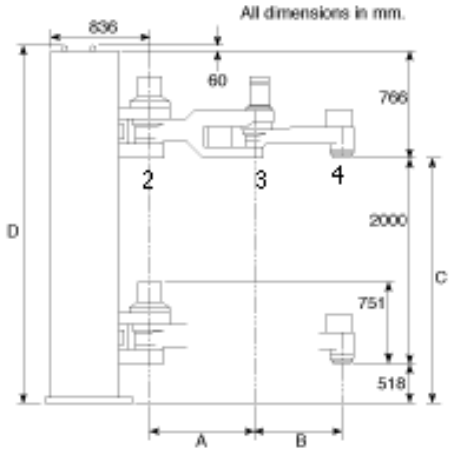
According to Equation (3),

$$M = 6(19 - 24) + 36 = 6$$

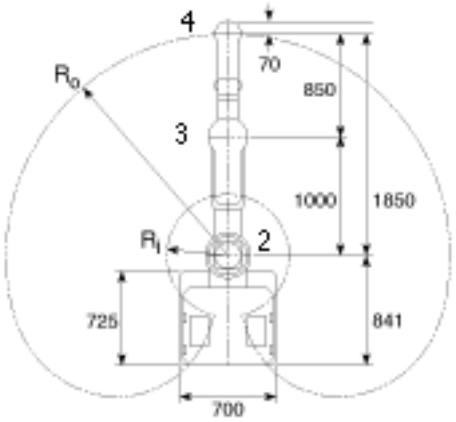
The Stewart Platform has actuators for all its six prismatic joints and it is therefore possible to control all six degrees of freedom.



(a) The Adept 1850 Palletizer



(b) side view (axes 2-4 are numbered)



(c) top view (axes 2-4 are numbered)

Figure 8 The Adept 1850 Palletizer

There are four degrees of freedom in this SCARA manipulator. Joint 1 is a sliding joint that carries the manipulator arm up or down. Joints 2-4 are rotary joints with vertical axes.

5.2 Geometry of planar robot manipulators

The mathematical modeling of spatial linkages is quite involved. It is useful to start with planar robots because the kinematics of planar mechanisms is generally much simpler to analyze. Also, planar examples illustrate the basic problems encountered in robot design, analysis and control without having to get too deeply involved in the mathematics. However, while the examples we will discuss will involve kinematic chains that are planar, all the definitions and ideas presented in this section are general and extend to the most general spatial mechanisms.

We will start with the example of the planar manipulator with three revolute joints. The manipulator is called a planar 3*R* manipulator. While there may not be any three degree of freedom (d.o.f.) industrial robots with this geometry, the planar 3*R* geometry can be found in many robot manipulators. For example, the shoulder swivel, elbow extension, and pitch of the Cincinnati Milacron T3 robot (Figure 2) can be described as a planar 3*R* chain. Similarly, in a four d.o.f. SCARA manipulator (Figure 8), if we ignore the prismatic joint for lowering or raising the gripper, the other three joints form a planar 3*R* chain. Thus, it is instructive to study the planar 3*R* manipulator as an example.

In order to specify the geometry of the planar 3*R* robot, we require three parameters, l_1, l_2, and l_3. These are the three link lengths. In Figure 9, the three joint angles are labeled θ_1, θ_2, and θ_3. These are obviously variable. The precise definitions for the link lengths and joint angles are as follows. For each pair of adjacent axes we can define a common normal or the perpendicular between the axes.

- The *ith* common normal is the perpendicular between the axes for joint *i* and joint *i*+1.
- The *ith* link length is the length of the *ith* common normal, or the distance between the axes for joint *i* and joint *i*+1.
- The *ith* joint angle is the angle between the (*i*-1)*th* common normal and *ith* common normal measured counter clockwise going from the (*i*-1)*th* common normal to the *ith* common normal.

Note that there is some ambiguity as far as the link length of the most distal link and the joint angle of the most proximal link are concerned. We define the link length of the most distal link from the most distal joint axis to a reference point or a tool point on the end effector[7]. Generally, this is the center of the gripper or the end point of the tool. Since there is no zeroth common normal, we measure the first joint angle from a convenient reference line. Here, we have chosen this to be the *x* axis of a conveniently defined fixed coordinate system.

[7]The reference point is often called the tool center point (TCP).

Another set of variables that is useful to define is the set of coordinates for the end effector. These coordinates define the position and orientation of the end effector. With a convenient choice of a reference point on the end effector, we can describe the position of the end effector using the coordinates of the reference point (x, y) and the orientation using the angle ϕ. The three end effector coordinates (x, y, ϕ) completely specify the position and orientation of the end effector[8].

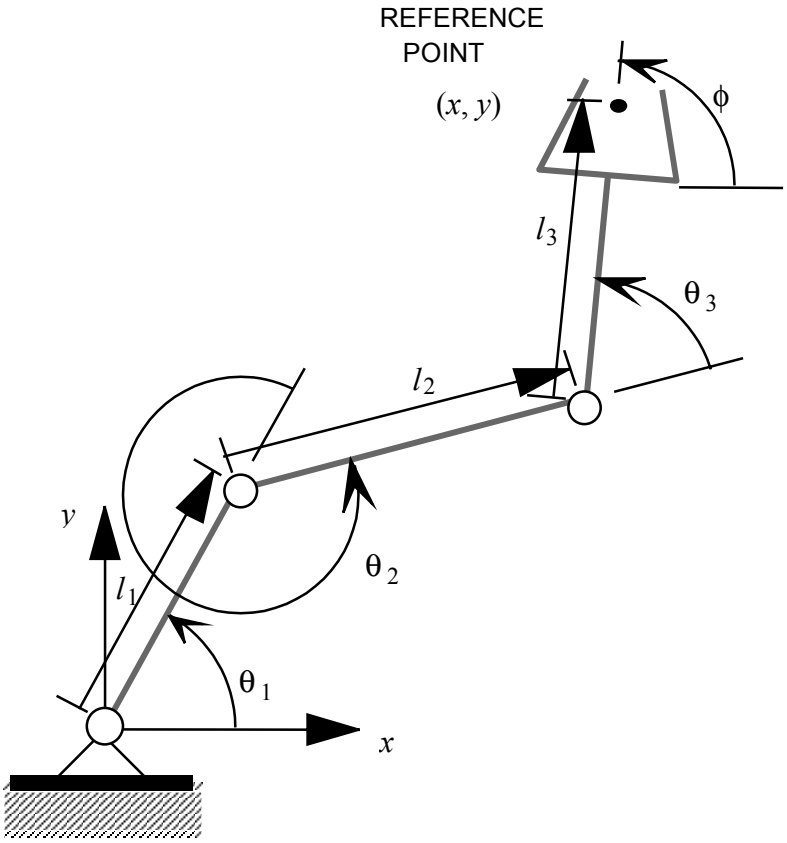


Figure 9 The joint variables and link lengths for a 3*R* planar manipulator

As another example, consider the three d.o.f. cylindrical robot in Figure 10. If we ignore the lift freedom, the rotation of the base and the extension of the arm give us the two d.o.f. robot shown in Figure 11 that we can call the *R*-*P* manipulator. It consists of a revolute joint and prismatic joint as shown in the figure. θ_1, the base rotation, and d_2, the arm extension, are the two joint variables. Note that there are no constant parameters such as the three link lengths in the 3*R* manipulator. The joint variable θ_1 is defined as before. Since there is no zeroth common normal,

[8]The description of the position and orientation of a three dimensional rigid body is significantly more complicated. For a spatial manipulator, a typical set of end effector coordinates would include three variables (x, y, z) for the position, and three Euler angles (θ, ϕ, ψ) for the orientation.

we measure the joint angle from the x axis which we have chosen to be horizontal. d_2 is defined as the distance from joint axis 1 to the reference point on the end effector. As in the previous example, the end effector coordinates are variables that completely specify the position and orientation of the end effector. In the figure, they are (x, y, ϕ).

Finally, we consider a Cartesian robot consisting of two prismatic joints at right angles. The *P-P* chain is found in x-y tables, plotters and milling machines. A schematic is shown in Figure 12. The simplest spatial manipulator is based on the *P-P-P* chain, which has a third prismatic joint. The three joint axes are mutually orthogonal. The Gantry robot in Figure 13 has this geometry. If you ignore the vertical up/down degree of freedom it is a *P-P* manipulator.

Figure 10 The RT3300 cylindrical robot (Seiko)

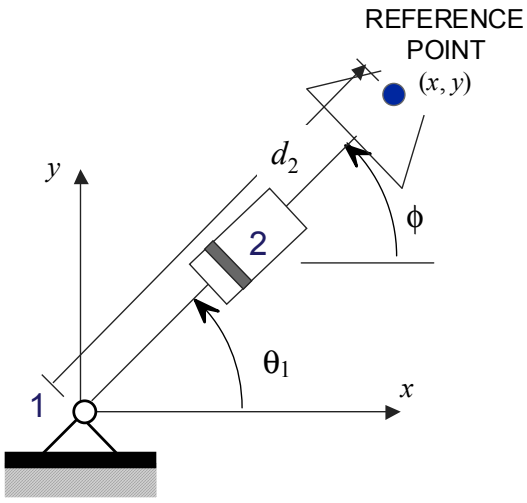


Figure 11 The joint variables and link lengths for a *R-P* planar manipulator

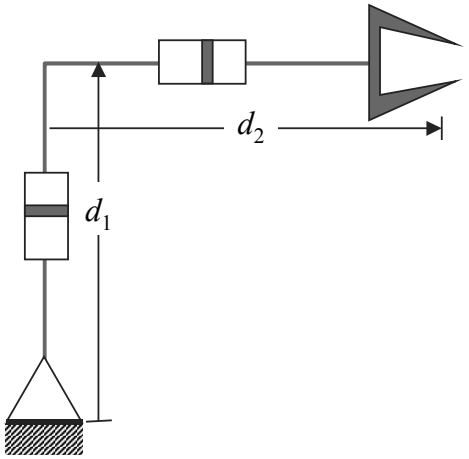
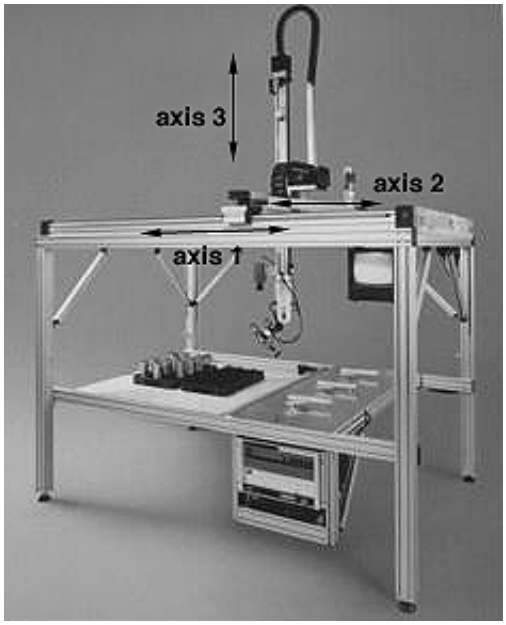


Figure 12 The joint variables for a *P-P* planar manipulator



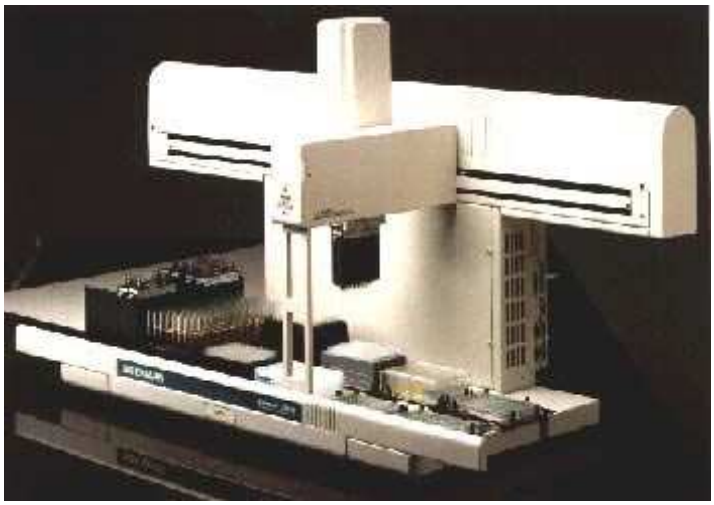


Figure 13 The G365 Gantry robot manipulator (CRS Robotics) on the left, and the Biomek 2000 Laboratory Automation Workstation (Beckman Coulter) on the right both have tooling mounted at the end of a *P-P-P* chain.

The end effector of a manipulator that has only prismatic joints is constrained to remain in the same orientation. Thus, the end effector coordinates for the *P-P* manipulator only include the coordinates of the reference point on the end effector (x, y).

In summary, in each case, we defined a set of constant parameters called *link lengths* (l_i) and set of joint variables or *joint coordinates* consisting of either joint angles (θ_i) or displacements

(d_i). We also defined a set of variables called *end effector coordinates*. The link lengths are constant parameters that define the geometry of the manipulator. The joint variables define the configuration of the manipulator by specifying the position of each joint. The end effector coordinates define the position and orientation of the end effector. If the joint coordinates specify the configuration of the manipulator, they should also specify the position and orientation of the end effector. Thus one should expect to find an explicit dependence of the end effector coordinates on the joint coordinates. Although it may not be obvious, there is also a dependence of the joint coordinates on the end effector coordinates. The next subsection will address this dependence and analyse the kinematics of robot manipulators.

5.3 Kinematic analysis of planar serial chains

Kinematics is the study of motion. In this subsection, we will explore the relationship between joint movements and end effector movements. More precisely, we will try to develop equations that will make explicit the dependence of end effector coordinates on joint coordinates and vice versa.

We will start with the example of the planar 3*R* manipulator. From basic trigonometry, the position and orientation of the end effector can be written in terms of the joint coordinates in the following way:

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ \phi &= (\theta_1 + \theta_2 + \theta_3) \end{aligned} \qquad (4)$$

Note that all the angles have been measured counter clockwise and the link lengths are assumed to be positive going from one joint axis to the immediately distal joint axis.

Equation (4) is a set of three nonlinear equations[9] that describe the relationship between end effector coordinates and joint coordinates. Notice that we have explicit equations for the end effector coordinates in terms of joint coordinates. However, to find the joint coordinates for a given set of end effector coordinates (x, y, ϕ), one needs to solve the nonlinear equations for θ_1, θ_2, and θ_3.

The kinematics of the planar *R*-*P* manipulator is easier to formulate. The equations are:

[9]The third equation is linear but the system of equations is nonlinear.

$$
\begin{aligned}
x &= d_2 \cos\theta_1 \\
y &= d_2 \sin\theta_1 \\
\phi &= \theta_1
\end{aligned}
\tag{5}
$$

Again the end effector coordinates are explicitly given in terms of the joint coordinates. However, since the equations are simpler (than in (4)), you would expect the algebra involved in solving for the joint coordinates in terms of the end effector coordinates to be easier. Notice that in contrast to (4), now there are three equations in only two joint coordinates, θ_1, and d_2. Thus, in general, we cannot solve for the joint coordinates for an arbitrary set of end effector coordinates. Said another way, the robot cannot, by moving its two joints, reach an arbitrary end effector position and orientation.

Let us instead consider only the position of the end effector described by (x, y), the coordinates of the end effector tool point or reference point . We have only two equations:

$$
\begin{aligned}
x &= d_2 \cos\theta_1 \\
y &= d_2 \sin\theta_1
\end{aligned}
\tag{6}
$$

Given the end effector coordinates (x, y), the joint variables can be computed to be:

$$
\begin{aligned}
d_2 &= +\sqrt{x^2 + y^2} \\
\theta_1 &= \tan^{-1}\left(\frac{y}{x}\right)
\end{aligned}
\tag{7}
$$

Notice that we restricted d_2 to positive values. A negative d_2 may be physically achieved by allowing the end effector reference point to pass through the origin of the x-y coordinate system over to another quadrant. In this case, we obtain another solution:

$$
\begin{aligned}
d_2 &= -\sqrt{x^2 + y^2} \\
\theta_1 &= \tan^{-1}\left(\frac{y}{x}\right)
\end{aligned}
\tag{8}
$$

In both cases (7-8), the inverse tangent function is multivalued[10]. In particular,

$$
\tan(x) = \tan(x + k\pi), \quad k = \ldots -2, -1, 0, 1, 2, \ldots \tag{9}
$$

However, if we limit θ_1 to the range $0 \le \theta_1 < 2\pi$, there is a unique value of θ_1 that is consistent with the given (x, y) and the computed d_2 (for which there are two choices).

[10]In Appendix 1, we define another inverse tangent function called *atan2* that takes two arguments, the sine and the cosine of an angle, and returns a unique angle in the range $[0, 2\pi)$.

The existence of multiple solutions is typical when we solve nonlinear equations. As we will see later, this poses some interesting questions when we consider the control of robot manipulators.

The planar Cartesian manipulator is trivial to analyze. The equations for kinematic analysis are:

$$x = d_2, \qquad y = d_1 \tag{10}$$

The simplicity of the kinematic equations makes the conversion from joint to end effector coordinates and back trivial. This is the reason why *P-P* chains are so popular in such automation equipment as robots, overhead cranes, and milling machines.

Direct kinematics

As seen earlier, there are two types of coordinates that are useful for describing the configuration of the system. If we focus our attention on the task and the end effector, we would prefer to use Cartesian coordinates or end effector coordinates. The set of all such coordinates is generally referred to as the *Cartesian space* or *end effector space*[11]**.** The other set of coordinates is the so called joint coordinates that is useful for describing the configuration of the mechanical lnkage. The set of all such coordinates is generally called the *joint space.*

In robotics, it is often necessary to be able to "map" joint coordinates to end effector coordinates. This map or the procedure used to obtain end effector coordinates from joint coordinates is called direct kinematics.

For example, for the 3-*R* manipulator, the procedure reduces to simply substituting the values for the joint angles in the equations

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ \phi &= (\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

and determining the Cartesian coordinates, x, y, and ϕ. For the other examples of open chains discussed so far (*R-P*, *P-P*) the process is even simpler (since the equations are simpler). In fact, for all serial chains (spatial chains included), the direct kinematics procedure is fairly straight forward.

On the other hand, the same procedure becomes more complicated if the mechanism contains one or more closed loops. In addition, the direct kinematics may yield more than one solution or

[11]Since each member of this set is an *n*-tuple, we can think of it as a vector and the space is really a vector space. But we shall not need this abstraction here.

no solution in such cases. For example, in the planar parallel manipulator in Figure 3, the joint positions or coordinates are the lengths of the three telescoping links (q_1, q_2, q_3) and the end effector coordinates (x, y, ϕ) are the position and orientation of the floating triangle. It can be shown that depending on the value of (q_1, q_2, q_3), the number of (real) solutions for (x, y, ϕ) can be anywhere from zero to six. For the Stewart Platform in Figure 4, this number has been shown to be anywhere from zero to forty.

5.4 Inverse kinematics

The analysis or procedure that is used to compute the joint coordinates for a given set of end effector coordinates is called inverse kinematics. Basically, this procedure involves solving a set of equations. However the equations are, in general, nonlinear and complex, and therefore, the inverse kinematics analysis can become quite involved. Also, as mentioned earlier, even if it is possible to solve the nonlinear equations, uniqueness is not guaranteed. There may not (and in general, will not) be a unique[12] set of joint coordinates for the given end effector coordinates.

We saw that for the *R-P* manipulator, the direct kinematics equations are:

$$\begin{aligned} x &= d_2 \cos\theta_1 \\ y &= d_2 \sin\theta_1 \end{aligned} \tag{6}$$

If we restrict the revolute joint to have a joint angle in the interval [0, 2π), there are two solutions for the inverse kinematics:

$$d_2 = \sigma\sqrt{x^2 + y^2}, \quad \theta_1 = \text{atan2}\left(\frac{y}{d_2}, \frac{x}{d_2}\right), \quad \sigma = \pm 1$$

Here we have used the *atan2* function in Appendix 1 to uniquely specify the joint angle θ_1. However, depending on the choice of σ, there are two solutions for d_2, and therefore for θ_1.

The inverse kinematics analysis for a planar 3-*R* manipulator appears to be complicated but we can derive analytical solutions. Recall that the direct kinematics equations (4) are:

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \tag{4a}$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \tag{4b}$$

$$\phi = (\theta_1 + \theta_2 + \theta_3) \tag{4c}$$

[12]The only case in which the analysis is trivial is the *P-P* manipulator. In this case, there is a unique solution for the inverse kinematics.

We assume that we are given the Cartesian coordinates, x, y, and ϕ and we want to find analytical expressions for the joint angles θ_1, θ_2, and θ_3 in terms of the Cartesian coordinates.

Substituting (4c) into (4a) and (4b) we can eliminate θ_3 so that we have two equations in θ_1 and θ_2:

$$x - l_3 \cos\phi = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \quad \text{(d)}$$

$$y - l_3 \sin\phi = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \quad \text{(e)}$$

where the unknowns have been grouped on the right hand side; the left hand side depends only on the end effector or Cartesian coordinates and are therefore known.

Rename the left hand sides, $x' = x - l_3 \cos\phi$, $y' = y - l_3 \sin\phi$, for convenience. We regroup terms in (d) and (e), square both sides in each equation and add them:

$$(x' - l_1 \cos\theta_1)^2 = (l_2 \cos(\theta_1 + \theta_2))^2$$

$$+$$

$$(y' - l_1 \sin\theta_1)^2 = (l_2 \sin(\theta_1 + \theta_2))^2$$

After rearranging the terms we get a single nonlinear equation in θ_1:

$$(-2l_1x')\cos\theta_1 + (-2l_1y')\sin\theta_1 + \left(x'^2 + y'^2 + l_1^2 - l_2^2\right) = 0 \quad \text{(f)}$$

Notice that we started with three nonlinear equations in three unknowns in (a-c). We reduced the problem to solving two nonlinear equations in two unknowns (d-e). And now we have simplified it further to solving a single nonlinear equation in one unknown (f).

Equation (f) is of the type

$$P\cos\alpha + Q\sin\alpha + R = 0 \quad \text{(g)}$$

Equations of this type can be solved using a simple substitution as shown in Appendix 2. There are two solutions for θ_1 given by:

$$\theta_1 = \gamma + \sigma\cos^{-1}\left(\frac{-\left(x'^2 + y'^2 + l_1^2 - l_2^2\right)}{2l_1\sqrt{x'^2 + y'^2}}\right) \quad \text{(h)}$$

where,

$$\gamma = \text{a tan 2}\left(\frac{-y'}{\sqrt{x'^2 + y'^2}}, \frac{x'}{\sqrt{x'^2 + y'^2}}\right),$$

and $\sigma = \pm 1$.

Note that there are two solutions for θ_1, one corresponding to σ=+1, the other corresponding to σ=-1. Substituting any one of these solutions back into Equations (d) and (e) gives us:

$$\cos(\theta_1 + \theta_2) = \frac{x' - l_1 \cos\theta_1}{l_2}$$

$$\sin(\theta_1 + \theta_2) = \frac{y' - l_1 \sin\theta_1}{l_2}$$

This allows us to solve for θ_2 using the atan2 function in Appendix 1:

$$\theta_2 = \text{atan2}\left(\frac{y' - l_1 \sin\theta_1}{l_2}, \frac{x' - l_1 \cos\theta_1}{l_2} \right) - \theta_1 \qquad \text{(i)}$$

Thus, for each solution for θ_1, there is one (unique) solution for θ_2.

Finally, θ_3 can be easily determined from (c):

$$\theta_3 = \phi - \theta_1 - \theta_2 \qquad \text{(j)}$$

Equations (h-j) are the inverse kinematics solution for the 3-*R* manipulator. For a given end effector position and orientation, there are two different ways of reaching it, each corresponding to a different value of σ. These different configurations are shown in Figure 14.

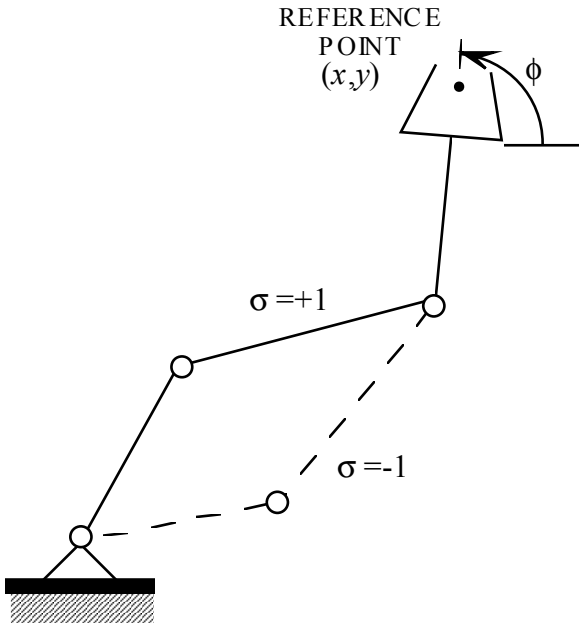


Figure 14 The two inverse kinematics solutions for the 3*R* manipulator: "elbow-up" configuration (σ=+1) and the "elbow-down" configuration (σ= -1)

Commanding a robot to move the end effector to a certain position and orientation is ambiguous because there are two configurations that the robot must choose from. From a

practical point of view, if the joint limits are such that one configuration cannot be reached this ambiguity is automatically resolved[13].

5.5 Velocity analysis

When controlling a robot to go from one position to another, it is not just enough to determine the joint and end effector coordinates of the target position. It may be necessary to continuously control the trajectory or the path taken by the robot as it moves toward the target position. This is essential to avoid obstacles in the workspace. More importantly, there are tasks where the trajectory of the end effector is critical. For example, when welding, it is necessary to maintain the tool at a desired orientation and a fixed distance away from the workpiece while moving uniformly[14] along a desired path. Thus one needs to control the velocity of the end effector or the tool during the motion. Since the control action occurs at the joints, it is only possible to control the joint velocities. Therefore, there is a need to be able to take the desired end effector velocities and calculate from them the joint velocities. All this requires a more detailed kinematic analysis, one that addresses velocities or the rate of change of coordinates in contrast to the previous section where we only looked at positions or coordinates.

Consider the $3R$ manipulator as an example. By differentiating Equation (4) with respect to time, it is possible to obtain equations that relate the the different velocities.

$$\dot{x} = -l_1\dot{\theta}_1 s_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2)s_{12} - l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)s_{123}$$
$$\dot{y} = l_1\dot{\theta}_1 c_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2)c_{12} + l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)c_{123}$$
$$\dot{\phi} = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

where we have used the short hand notation:

$$s_1 = \sin\theta_1, \quad s_{12} = \sin(\theta_1 + \theta_2), \quad s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$$
$$c_1 = \cos\theta_1, \quad c_{12} = \cos(\theta_1 + \theta_2), \quad c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$$

$\dot{\theta}_i$ denotes the joint speed for the *ith* joint or the time derivative of the *ith* joint angles, and $\dot{x}$, $\dot{y}$, and $\dot{\phi}$ are the time derivatives of the end effector coordinates. Rearranging the terms, we can write this equation in matrix form:

---

[13]This is true of the human arm. If you consider planar movements, because the human elbow cannot be hyper extended, there is a unique solution for the inverse kinematics. Thus the central nervous system does not have to worry about which configuration to adopt for a reaching task.

[14]In some cases, a weaving motion is required and the trajectory of the tool is more complicated.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12} + l_3 s_{123}) & -(l_2 s_{12} + l_3 s_{123}) & -l_3 s_{123} \\ (l_1 c_1 + l_2 c_{12} + l_3 c_{123}) & (l_2 c_{12} + l_3 c_{123}) & l_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{11}$$

The 3×3 matrix is called the Jacobian matrix[15] and we will denote it by the symbol **J**. If you look at the elements of the matrix they express the rate of change of the end effector coordinates with respect to the joint coordinates:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial \phi}{\partial \theta_1} & \frac{\partial \phi}{\partial \theta_2} & \frac{\partial \phi}{\partial \theta_3} \end{bmatrix}$$

Given the rate at which the joints are changing, or the vector of joint velocities,

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix},$$

using Equation (11), we can obtain expressions for the end effector velocities,

$$\dot{\mathbf{p}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}.$$

If the Jacobian matrix is non singular (its determinant is non zero and the matrix is invertible), then we can get the following expression for the joint velocities in terms of the end effector velocities:

$$\dot{\mathbf{p}} = \mathbf{J}\dot{\mathbf{q}}, \qquad \dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{p}} \tag{12}$$

Thus if the task (for example, welding) is specified in terms of a desired end effector velocity, Equation (12) can be used to compute the desired joint velocity provided the Jacobian is non singular:

[15]The name Jacobian comes from the terminology used in multi-dimensional calculus.

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}^{des} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12} + l_3 s_{123}) & -(l_2 s_{12} + l_3 s_{123}) & -l_3 s_{123} \\ (l_1 c_1 + l_2 c_{12} + l_3 c_{123}) & (l_2 c_{12} + l_3 c_{123}) & l_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}^{des}$$

Naturally we want to determine the conditions under which the Jacobian becomes singular. This can be done by computing the determinant of **J** and setting it to zero. Fortunately, the expression for the determinant of the Jacobian, in this example, can be simplified using trigonometric identities to:

$$|\mathbf{J}| = l_1 l_2 \sin \theta_2 \qquad (13)$$

This means that the Jacobian is singular only when θ_2 is either 0 or 180 degrees. Physically, this corresponds to the elbow being completely extended or completely flexed. Thus, as long we avoid going through this configuration, the robot will be able to follow any desired end effector velocity.

5.6 Appendix

5.6.1 The ambiguity in inverse trigonometric functions

Inverse trigonometric functions have multiple values. Even within a 360 degree range they have two values. For example, if

$$y = \sin x$$

the inverse sin function gives two values in a 360 degree interval:

$$\sin^{-1} y = x, \pi\text{-}x$$

Of course we can add or subtract 2π from either of these solutions and obtain another solution.

This is true of the inverse cosine and inverse tangent functions as well. If

$$y = \cos x,$$

the inverse cosine function yields:

$$\cos^{-1} y = x, -x$$

Similarly, for the tangent function

$$y = \tan x,$$

the inverse tangent function yields:

$$\tan^{-1} y = x, \pi + x$$

This multiplicity is particularly troublesome in robot control where an ambiguity may mean that there is more than one way of reaching a desired position (see discussion on inverse kinematics). This problem is circumvented by defining the *atan2* function which requires two arguments and returns a unique answer in a 360□ range.

The atan2 function takes as arguments the sine and cosine of a number and returns the number. Thus if ,

$$s = \sin x; \quad c = \cos x$$

the atan2 function takes s and c as the argument and returns x:

$$\text{atan2}\,(s, c) = x$$

The main idea is that the additional information provided by the second argument eliminates the ambiguity in solving for x. To see this consider the simple problem where we are given:

$$s = \frac{1}{2}; \quad c = \frac{\sqrt{3}}{2}$$

and we are required to solve for x. If we use the inverse sine function and restrict the answer to be in the interval $[0, 2\pi)$, we get the result:

$$x = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$$

Since we know the cosine to be $\frac{\sqrt{3}}{2}$, we can quickly verify by taking cosines of both candidate solutions that the first solution is correct and the second one is incorrect.

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}; \quad \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

The atan2 function goes through a similar algorithm to figure out a unique solution in the range $[0,2\pi)$.

$$\text{atan2}\,(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{\pi}{6}$$

The atan2 function is a standard function in most C, Pascal and Fortran compilers.

5.6.2 Solution of the nonlinear equation in (g)

$$P\cos\alpha + Q\sin\alpha + R = 0 \qquad \text{(g)}$$

Define γ so that

$$\cos\gamma = \frac{P}{\sqrt{P^2+Q^2}} \quad \text{and} \quad \sin\gamma = \frac{Q}{\sqrt{P^2+Q^2}}$$

Note that this is always possible. γ can be determined by using the atan2 function:

$$\gamma = \text{a}\tan 2\left(\frac{Q}{\sqrt{P^2+Q^2}}, \frac{P}{\sqrt{P^2+Q^2}}\right)$$

Now (g) can be rewritten as:

$$\cos\gamma\cos\alpha + \sin\gamma\sin\alpha + \frac{R}{\sqrt{P^2+Q^2}} = 0$$

or

$$\cos(\alpha-\gamma) = \frac{-R}{\sqrt{P^2+Q^2}}$$

This gives us two solutions for α in terms of the known angle γ:

$$\alpha = \gamma + \sigma\cos^{-1}\left(\frac{-R}{\sqrt{P^2+Q^2}}\right), \sigma = \pm 1$$