An Alternative Algebra for Deriving Equations of Motion of Flexible Manipulators

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Received February 25, 1988; accepted July 25, 1988

An alternative means for deriving equations of motion of complex systems is demonstrated. Since the method is energy based, it is useful for elastic systems. Because the method can handle vectors expressed relative to rotating coordinate systems, it does not require the introduction of coordinate transformations and thereby produces equations in a simple form. The article shows that Kane's method for rigid body systems is a special case of this alternative method. Two example problems show how the algebra can be applied to rigid and flexible nonholonomic systems.

本論文では、複雑なシステムの運動方程式を導出するための二者択一法が証明される。この方法はエネルギに基づくため、弾性系に適用可能である。またこれは回転座標系について表現されたベクトルを扱えるため座標変換の必要がない。その結果、方程式を簡潔な形式で生成することができる。本報では剛体系についてのKaneの方法が、二者択一法の特殊な場合であることを示す。２つの例題が、この代数がどのようにして剛体及び弾性非ホロノーム系に適用できるかを示す。

INTRODUCTION

Many research applications require the dynamic equations of motion for manipulators. In complex manipulator-environment systems, it is necessary to simulate realistically the motion of the manipulator in its environment. Typical examples are Space Shuttle flight simulations performed by NASA. Similarly, simulation aids the design of complex systems. Although some simulation applications can be performed off-line, it is sometimes necessary to simulate motion in real time, as when simulation is a training aid for human operators.

Journal of Robotic Systems, 5(6), 553–566 (1988)
 CCC 0741-2223/88/060553-14$4.00

Another application of dynamic equations is the control of manipulators. Many control schemes have been proposed for increasing the performance of manipulators that require the computation of driving torque. Driving torque must be computed by the dynamic equations. Many of the control algorithms require torque to be computed in real time.

There have been many attempts to solve manipulator dynamic equations in real time. Some researchers ignore terms such as the Coriolis and Centripetal accelerations. Ignoring terms has the advantage of reducing the number of computations but the disadvantage of being approximate. The approximation may be unsatisfactory for some applications.

Another method for solving dynamic equations in real time is parallel processing, a satisfactory solution for some cases, but excessive hardware expenditure for others.

Recursive calculation of dynamic equations reduces the computation time. Recursion, a simple process that utilizes the open chain structure of a manipulator to simplify computation, may not be applicable to systems that are not open chain structures; hence its applicability is limited.

Closed form reductions reduce the numerical complexity of solving dynamic equations. Closed form reductions usually require considerable experience and stamina. The effort required for closed form reduction can vary greatly among the methods used for deriving the equations. For example, a method that produces equations in terms of matrix multiplications (such as the standard Lagrange method) is difficult to reduce. A technique that produces equations in terms of vector operations like dot and cross products is much easier to reduce than the matrix formulation. Either method of derivation produces the same equations, but each can produce the equations in different initial forms.

As an example of this concept, consider some of the techniques available for deriving equations of rigid, open chain manipulators. One technique is to use Lagrange's equations.[1] The method of Lagrange begins by expressing the scalar energy quantities of a system. Since kinetic energy is a function of absolute velocity, the presence of multiple rigid bodies makes it necessary to introduce coordinate transformations. These transformations are complicated trigonometric functions. After introducing the transformations, several partial and total derivatives are computed. The trigonometry makes it difficult (if not impossible) to simplify the equations. One significant advantage of the Lagrange technique is in not including the forces that maintain system constraints.

Another popular technique utilizes Newton's equations. Although the method can be derived from Lagrange's equations, Newton's equations are distinctive. The method is usually based on vector equations and requires the computation of absolute acceleration. Unlike Lagrange's method, the technique need not introduce coordinate transformations at the outset; hence the results are often expressed with vector dot and cross products. This vector form allows reduction by utilizing the special properties of vector products. A major disadvantage of the technique is that even constraint forces must be included.

A third technique popular in the study of spacecraft dynamics is the method of Thomas Kane.[2] Kane's method (also known as Lagrange's form of D'Alembert's Principle) is a vector formulation based on a modification of Newton's equations. Kane's method is not energy based and requires the computation of absolute velocity and acceleration as in Newton's method. Since the resulting equations are in vector form, so they can be reduced easily. One advantage of the method is that forces maintaining constraints do not enter; therefore there are fewer unknowns than in Newton's method. The method's utility in deriving simplified dynamic equations was demonstrated recently.[3] Although at first glance the method appears awkward, it is powerful in the hands of an experienced user.

In the area of flexible (compliant) manipulators, many techniques are energy based and are not as well developed as those for rigid systems. As expected, the energy techniques deal with scalars and require introduction of coordinate transformations before computation of partial and total derivatives. Two advantages of energy methods are that boundary conditions arise in the course of equation derivation and forces maintaining constraints are unimportant.

Vector-based Newton's equation techniques have been used in the derivation of equations for flexible systems. Application of Newton's equations require one to sum forces (stress) acting on an infinitesimal piece of the flexible system. For many researchers working in the area of flexible systems this is an unnatural process.

A method follows for formulating equations of motion of flexible, open chain manipulators using energy. The method differs from present techniques in that the formulation does not require the introduction of coordinate transformations. This is made possible through the proper modification of the partial and total derivatives normally required. Because the method is energy based, it should be natural for many researchers. The required boundary conditions arise during the derivation, and forces maintaining constraints do not enter. Because the method is vector based, the resulting equations should be easy to simplify. The method is applicable to six-degree-of-freedom manipulators, which allows it to be used for the control and/or simulation of practical manipulators.

EQUIVALENCE OF ENERGY AND KANE'S EQUATIONS

The technique demonstrated in this article begins with energy concepts and derives equations of motion similar to those of Kane's method. The equivalence between these techniques is not new. It has been shown[2] that the Passerello–Huston equations can be used to compute Kane's generalized inertia forces given kinetic energy.

Consider a system with n degrees of freedom. It is often possible to define n variables q_i that satisfy all holonomic constraints. The q_i are called generalized coordinates. The velocity of point P belonging to the system can be expressed as a function of q_i, $\dot{q}_i$ and time t, where $\dot{q}_i$ denotes the time derivative of q_i. It

may be convenient to define n quantities u_i, which are functions of q_i and $\dot{q}_i$, as:

$$\dot{q}_s = \sum_{r=1}^{n} W_{sr} u_r + X_s \quad \text{for} \quad s = 1, \ldots, n. \tag{1}$$

The u_i are called generalized speeds.[2] It must be possible to solve Eq. 1 for the $\dot{q}_i$ terms. This means that matrix W is nonsingular.

If m simple nonholonomic constraints are applied to the system, only p ($p = n - m$) of the $\dot{q}_i$ and u_i are independent of each other. These constraints can be expressed as:

$$u_r = \sum_{s=1}^{p} A_{rs} u_s + B_r \quad \text{for } r = p+1, \ldots, n. \tag{2}$$

Once the p independent u_i are known (from the equations of motion), the relations between u_i and $\dot{q}_i$ and the nonholonomic constraints allow solution (integration) for the n values of q_i. The quantities A, B, W, and X are functions of time and the n coordinates q.

The Passerello–Huston equations for a simple nonholonomic system defined above are:

$$F_r^* = \sum_{s=1}^{n} \left(\frac{d}{dt} \frac{\partial KE}{\partial \dot{q}_s} - \frac{\partial KE}{\partial q_s} \right) \left(W_{sr} + \sum_{k=p+1}^{n} W_{sk} A_{kr} \right) \tag{3}$$

where F_r^* is Kane's generalized inertia force for generalized speed u_r, and KE is kinetic energy. The connection between potential energy and Kane's generalized active force has also been established.[2]

With the Passerello–Huston Eq. 3 one can begin with energy terms and arrive at Kane's equations. There are, however, several practical problems to consider. First, because Eq. 3 requires the kinetic energy to be expressed as a scalar function, coordinate transformations must be introduced immediately. This makes the algebra required to compute the partial and total derivatives excessive. Second, expressing the energy in terms of q rather than u erases some of the advantages afforded by Kane's method.

The disadvantages are so significant that although Kane's method does have an energy base, there is little merit in utilizing the relationships.[2] In the remainder of the article, a method will be demonstrated that allows manipulation of energy terms without the serious shortcomings of Eq. 3.

VARIATIONS OF VECTOR QUANTITIES

The present method is based on Hamilton's principle, which states that the variation of the time derivative of the Lagrangian is zero. To implement the technique, it is necessary to compute the variation of energy. Ordinarily the

operation of variation is defined only for scalars (or matrices), but derivation of equations without the introduction of coordinate transformation requires computation of variations of vectors expressed in rotating coordinate systems. Some of the basic definitions have been published[4] and these are reviewed.

Variation of a Vector

To compute what is herein defined as the absolute variation of a vector, it is necessary to express the vector with components along three noncoplanar, stationary coordinate directions and compute the variation in each scalar component. This can be expressed as:

$$^{N}\delta\vec{V} = \sum_{i=1}^{3} \delta(f_i)\vec{n}_i \tag{4}$$

where $\vec{n}_i$ are fixed noncoplanar unit vectors in inertial reference frame n, and f_i are scalar functions.

In the current techniques that use vector rotations, whenever a variation is computed the vector terms are first expressed relative to an inertial reference. This requires the introduction of coordinate transforms *before* variation.

Relative Variation

Similarly, relative variations are computed as the variation in the scalar components of a vector expressed relative to a noninertial reference. Mathematically this is

$$^{R}\delta\vec{V} = \sum_{i=1}^{3} \delta(g_i)\vec{r}_i \tag{5}$$

where reference R is noninertial and g_i are scalar functions.

Relative and absolute variations differ. Relative variations are often easier to compute, but Hamilton's principle requires computing absolute variations.

Relating Variations

It is possible to relate relative and absolute variations.[4,5] If coordinate system R is rotating relative to frame N, the relative variations in these coordinate systems are related as:

$$^{N}\delta\vec{V} = {}^{R}\delta\vec{V} + {}^{N}\vec{\theta}^{R} \times \vec{V} \tag{6}$$

$^{N}\vec{\theta}^{R}$ is a vector of the possible rotations frame R can possess relative to frame N. Kane introduced vector $^{N}\vec{\theta}^{R}$ when discussing compatible virtual displacements[5] but did not expound use of $^{N}\vec{\theta}^{R}$ for solving practical problems.

Time Differentiation of Variations

The operations of relative differentiation and variation commute only when computed relative to the same reference.[4] Consider a vector $\vec{V}$ expressed in coordinate system C. First compute the time derivative relative to a system B and follow with the variation relative to a system A:

$$
\begin{aligned}
{}^{A}\delta\,{}^{B}D\vec{V} &= {}^{A}\delta({}^{A}D\vec{V} + {}^{B}\vec{\omega}^{A} \times \vec{V}) \\
&= {}^{A}D^{A}\delta\vec{V} + ({}^{A}\delta\,{}^{B}\vec{\omega}^{A}) \times \vec{V} + {}^{B}\vec{\omega}^{A} \times {}^{A}\delta\vec{V} \qquad (7) \\
&= {}^{B}D^{A}\delta\vec{V} + ({}^{A}\delta\,{}^{B}\vec{\omega}^{A}) \times \vec{V}.
\end{aligned}
$$

The operator ${}^{i}D$ represents differentiation relative to frame i, [illegible] is angular velocity of frame A as seen in B.

By reversing the order of application of the operators, Eq. 7 can be expressed as:

$$
\begin{aligned}
{}^{B}D^{A}\delta\vec{V} &= {}^{B}D({}^{B}\delta\vec{V} + {}^{A}\vec{\theta}^{B} \times \vec{V}) \\
&= {}^{B}\delta\,{}^{B}D\vec{V} + ({}^{B}D^{A}\vec{\theta}^{B}) \times \vec{V} + {}^{A}\vec{\theta}^{B} \times {}^{B}D\vec{V} \qquad (8) \\
&= {}^{A}\delta\,{}^{B}D\vec{V} + ({}^{B}D^{A}\vec{\theta}^{B}) \times \vec{V}.
\end{aligned}
$$

Combining Eqs. 7 and 8 and using ${}^{B}\vec{\omega}^{A} = -{}^{A}\vec{\theta}^{B}$ yields:[4]

$$
{}^{B}D\,{}^{B}\vec{\theta}^{A} = {}^{A}\delta\,{}^{B}\vec{\omega}^{A}. \qquad (9)
$$

Variation of a Dyadic

A dyadic $\vec{\vec{I}}$ is defined as:

$$
\vec{\vec{I}} = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\vec{a}_i\vec{b}_j \qquad (10)
$$

where I_{ij} is a scalar and $\vec{a}_i$ and $\vec{b}_j$ are noncoplanar unit vectors fixed in coordinate systems A and B. Vectors $\vec{A}$ and $\vec{B}$ are normally not commutative. Dyadics have special properties[5] but can be thought of as vectors whose components are vectors. Dot and cross product are defined for dyadics, but one must refer to the operators as operating on the left or the right. Consider the variation relative to frame C of a dyadic, expressed as:

$$
{}^{C}\delta\vec{\vec{I}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \{[{}^{C}\delta(I_{ij}\vec{a}_i)]\vec{b}_j + (I_{ij}\vec{a}_i)({}^{C}\delta\vec{b}_j)\}. \qquad (11)
$$

The variation is computed by treating the dyadic as a vector whose components are also vectors. From Eq. 6, Eq. 11 becomes:

$$^{C}\delta\vec{\vec{I}} = \sum_{i=1}^{3}\sum_{j=1}^{3}\{[^{A}\delta(I_{ij}\vec{a}_i) + {}^{C}\vec{\theta}^{A}\times(I_{ij}\vec{a}_i)]\vec{b}_j + (I_{ij}\vec{a}_i)[^{B}\delta\vec{b}_j + {}^{C}\vec{\theta}^{B}\times\vec{b}_j]\} \quad (12)$$

which reduces to:

$$^{C}\delta\vec{\vec{I}} = \sum_{i=1}^{3}\sum_{j=1}^{3}\{(\delta I_{ij})\vec{a}_i\vec{b}_j + {}^{C}\vec{\theta}^{A}\times I_{ij}\vec{a}_i\vec{b}_j - I_{ij}\vec{a}_i(\vec{b}_j\times{}^{C}\vec{\theta}^{B})\} \quad (13)$$

and finally becomes:

$$^{C}\delta\vec{\vec{I}} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}(\delta I_{ij})\vec{a}_i\vec{b}_j\right) + {}^{C}\vec{\theta}^{A}\times\vec{\vec{I}} - \vec{\vec{I}}\times{}^{C}\vec{\theta}^{B}. \quad (14)$$

Computing the Total Variation of Vectors

Let a position vector $\vec{p}$ relative to coordinate system A in a system with n generalized coordinates be expressed as:

$$\vec{p} = \sum_{i=1}^{3} f_i\vec{a}_i \quad (15)$$

where $\vec{a}_i$ are unit vectors fixed in coordinate system A, and f_i are scalar functions of the n generalized coordinates and time. The variation of $\vec{p}$ relative to frame A is:

coordenadas generalizadas

$$^{A}\delta\vec{p} = \sum_{r=1}^{n}\frac{\partial\vec{p}}{\partial q_r}(\delta q_r). \quad (16)$$

If n generalized speeds are defined as in Eq. 1, the derivative of $\vec{p}$ in frame A (the velocity of P in A) is:

$$\vec{V} = \sum_{r=1}^{n}\frac{\partial\vec{p}}{\partial q_r}\left(\sum_{s=1}^{n} W_{rs}u_s + X_r\right) + \frac{\partial\vec{p}}{\partial t}. \quad (17)$$

If there are no nonholonomic constraints, all δq_r in Eq. 16 can be considered arbitrary; hence another set of arbitrary variations δu_s can be defined as:

$$\delta q_r = \sum_{s=1}^{n} W_{rs}\delta u_s \quad (18)$$

because W is nonsingular. Equation 16 can be written;

$$^{A}\delta\vec{p} = \sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\partial\vec{p}}{\partial q_r}W_{rs}\delta u_s = \sum_{s=1}^{n}{}^{A}\vec{V}^{P}_{u_s}\delta u_s \tag{19}$$

where $^{A}\vec{V}^{P}_{u_s}$ is Kane's partial velocity of point P in frame A for generalized speed u_s. This last substitution can be verified by Eq. 17.

Now consider the case of m simple nonholonomic constraints given by Eq. 2. By substituting Eq. 2 into 17, velocity can be expressed:

$$\begin{aligned}\vec{V} = &\sum_{r=1}^{n}\sum_{s=1}^{p}\frac{\partial\vec{p}}{\partial q_r}\left[W_{rs} + \left(\sum_{j=p+1}^{n}W_{rj}A_{js}\right)\right]u_s + \\ &\sum_{r=1}^{n}\frac{\partial\vec{p}}{\partial q_r}\left(\sum_{j=p+1}^{n}W_{rj}B_j + X_r\right) + \frac{\partial\vec{p}}{\partial t}.\end{aligned} \tag{20}$$

The variation of Eq. 3 can be expressed as:

$$\delta u_r = \sum_{s=1}^{p}A_{rs}\delta u_s \quad \text{for } r = p+1, \ldots, n. \tag{21}$$

Notice that the vector B does not contribute to the variation because the variation is taken instantaneously. Since B is not a function of the variables u_r, it has no variation. This concept is discussed in Lanczos[1] under the topic of rheonomic nonholonomic constraints. Note that although B disappears from Eq. 21, it does contribute to the problem. The equations of motion are expressed as p equations that can be integrated for the p independent u_r ($r = 1, \ldots, p$). Equations 3 (which contain B) are used to solve for the remaining u_s, then Eq. 2 are integrated for q_r.

Substituting Eq. 21 into Eq. 19 results in:

$$\begin{aligned}^{A}\delta\vec{p} &= \sum_{r=1}^{n}\sum_{s=1}^{p}\frac{\partial\vec{p}}{\partial q_r}\left[W_{rs} + \left(\sum_{j=p+1}^{n}W_{rj}A_{js}\right)\right]\delta u_s \\ &= \sum_{r=1}^{n}{}^{A}\vec{V}^{P}_{u_r}\delta u_r\end{aligned} \tag{22}$$

The term $^{A}\vec{V}^{P}_{u_r}$ is Kane's nonholonomic partial velocity as demonstrated in Eq. 20.

These results can be summarized as follows. Suppose the time derivative of a position vector $\vec{p}$ belonging to a system with p independent generalized speed u_i is written as a function of the u_i and time as $\vec{V}(u_1, \ldots, u_p, t)$, and the absolute variation of the vector $\vec{p}$ (the virtual displacement) is required. The virtual displacement must be an expression of all possible changes the vector can experience independent of time in light of all constraints (holonomic and

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nonholonomic) imposed on the system. This can be expressed as:

$$^{A}\delta\vec{r} = \sum_{i=1}^{p} \frac{\partial \vec{V}}{\partial u_i} \delta_i \tag{23}$$

where δ_i represents an arbitrary virtual quantity and $\vec{V}$ is defined in Eq. 17 for holonomic systems and in Eq. 20 for a nonholonomic system.

The quantities $^{A}\vec{\theta}^{B}$ can be interpreted similarly. Suppose the angular velocity of frame B relative to A is expressed as $^{A}\vec{\omega}^{B}(u_1, \ldots, u_p, t)$. Since $^{A}\vec{\theta}^{B}$ is an expression of all possible rotations of frame B relative to A, it can be written as:

$$^{A}\vec{\theta}^{B} = \sum_{i=1}^{p} \frac{\partial {}^{A}\vec{\omega}^{B}}{\partial u_i} \delta_i. \tag{24}$$

Note that the quantities δ_i found in Eqs. 23 and 24 are identical. The formal derivation of Eq. 24 is performed similarly to that of Eq. 23.

Example Problems

Two examples demonstrate the concept discussed in this article, a simpler example can be found in Ref. 4. The method can be applied to complex systems such as a six-degree-of-freedom robot in a similar manner. First consider the problem of deriving the equations of motion of a three-dimensional, n link, open-loop manipulator. Let initial reference frame 0 be attached to the ground and moving frames 1 through n be attached to each of the manipulator links.

Through proper definition of the inertia dyadic,[2] the kinetic energy of the system is:

$$KE = \frac{1}{2}\left\{\sum_{i=1}^{n} [{}^{0}\vec{\omega}^{i} \cdot \bar{\bar{I}}_i \cdot {}^{0}\vec{\omega}^{i} + m_i {}^{0}\vec{V}^{i^*} \cdot {}^{0}\vec{V}^{i^*}]\right\} \tag{25}$$

where $\bar{\bar{I}}_i$ is the inertia dyadic for body i, m_i is mass of body i, i^* is the mass center of body i, and ${}^{0}\vec{V}^{i^*}$ is absolute velocity. Assuming only gravity loading, the potential energy is:

$$PE = -\sum_{i=1}^{n} (\vec{g} \cdot \vec{p}^{i^*}) \tag{26}$$

$\vec{p}^{i^*}$ is the position of the mass center of body i, and $\vec{g}$ is the gravity vector. Subtracting potential from kinetic and integrating, the functional in Hamilton's principle is obtained:

$$f = \int_t \left[\sum_{i=1}^{n} \frac{1}{2}({}^{0}\vec{\omega}^{i} \cdot \bar{\bar{I}}_i \cdot {}^{0}\vec{\omega}^{i} + m_i {}^{0}\vec{V}^{i^*} \cdot {}^{0}\vec{V}^{i^*}) + (\vec{g} \cdot \vec{p}^{i^*})\right] dt. \tag{27}$$

The equations of motion are obtained by setting all arbitrary variations of the functional to zero, hence the variation of f must be computed. One way to do this is:

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$$\delta f = \int_t \left\{ \sum_{i=1}^{n} [{}^0\vec{\omega}^i \cdot \bar{\bar{I}}_i \cdot {}^i\delta^0\vec{\omega}^i + m_i {}^0\vec{V}^{i*} \cdot {}^0\delta^0\vec{V}^{i*} + ({}^i\delta\vec{g}) \cdot \vec{p}^{i*} + \vec{g} \cdot ({}^i\delta\vec{p}^{i*})] \right\} dt = 0. \tag{28}$$

Since the energy terms are scalars, the variations in all frames are identical. For example ${}^0\delta({}^0\vec{V}^{i*} \cdot {}^0\vec{V}^{i*})$ is the same as ${}^i\delta({}^0\vec{V}^{i*} \cdot {}^0\vec{V}^{i*})$. Distributing the variation among the dotted terms requires careful observation of the frame of the variation since the individual terms are vectors. In Eq. 28 the first term is varied in frame i because the inertia dyadic is constant relative to frame i; therefore, only the angular velocity vector contributes to the variation. The linear velocity term in Eq. 28 is varied in the inertial frame. The potential energy term is varied in some arbitrarily selected frame j. When the variation is applied to the gravity vector, there is a contribution because the vector is not fixed in frame j.

The next step is to reduce all variations of derivative terms into derivatives of variations by some of the identities in the last section. Equation 28 can be expressed as:

$$\delta f = \int_t \left\{ \sum_{i=1}^{n} [{}^0\vec{\omega}^i \cdot \bar{\bar{I}}_i \cdot {}^0D^0\vec{\theta}^i + m_i {}^0\vec{V}^{i*} \cdot {}^0D^0\delta\vec{p}^{i*} + ({}^j\vec{\theta}^0 \times \vec{g}) \cdot \vec{p}^{i*} + \vec{g} \cdot ({}^j\delta\vec{p}^{i*})] \right\} dt = 0.$$

Using Green's theorem, Eq. 29 can be written:

$$\delta f = \int_t \left\{ \sum_{i=1}^{n} [-{}^0D({}^0\vec{\omega}^i \cdot \bar{\bar{I}}_i) \cdot {}^0\vec{\theta}^i - {}^0D(m_i {}^0\vec{V}^{i*}) \cdot {}^0\delta\vec{p}^{i*} + (\vec{p}^{i*} \times \vec{g}) \cdot {}^0\vec{\theta}^j + \vec{g} \cdot ({}^j\delta\vec{p}^{i*})] \right\} dt = 0. \tag{30}$$

Individual equations (n of them) are extracted from Eq. 30 by expressing the variations as a function of n arbitrary virtual quantities. For example, with n quantities u_j, as discussed in the last section, so that linear and angular velocities of the system can be expressed as functions of position, time and u_j, ${}^0\vec{\theta}^i$ can be expressed as:

$${}^0\vec{\theta}^i = \sum_{j=1}^{n} \frac{\partial {}^0\vec{\omega}^i}{\partial u_j} \delta u_j. \tag{31}$$

The quantity ${}^j\delta\vec{p}^{i*}$ is equal to:

$${}^j\delta\vec{p}^{i*} = \sum_{j=1}^{n} \frac{\partial({}^jD\vec{p}^{i*})}{\partial u_j} \delta u_j \tag{32}$$

If the u_j are chosen to be the n joint variables, it is easy to compute the terms in Eqs. 31 and 32. This choice is not necessary, however, and significant simplifications can come with an alternative.[3] Once the variations are expressed in terms of the n quantities u_j, equations are extracted by collecting all terms multiplying u_j and setting the collection to zero for all j.

The resulting equations are similar to those from Kane's technique. Because the terms appear in a high level form involving dot and cross products, it is possible to manipulate the terms into significantly different forms before any coordinate transformations need to be computed. Significant differences from Kane's method are the researcher's advantage of manipulating the variation in different coordinate systems to find simplifications, and the method being energy based, thereby compatible with many other techniques for elastic systems.

Kinetic Energy Terms for a Flexible Body

This example considers the kinetic energy terms for a system of several flexible bodies. The potential energy is simple to compute. Since the total energy is merely the sum of energy of all bodies, consider the energy contribution of a single body B. As shown in Figure 1, let $\vec{p}^P$ represent the absolute position of a particle belonging to B. The time integral of the kinetic energy of B can be expressed as

$$\int_t KEdt = \int_t \int_{vol} \frac{1}{2} \rho \, {}^A D\vec{p}^P \cdot {}^A D\vec{p}^P d_{vol} dt \tag{33}$$

where A is an inertial reference frame.

The problem becomes complicated when the variation of kinetic energy is

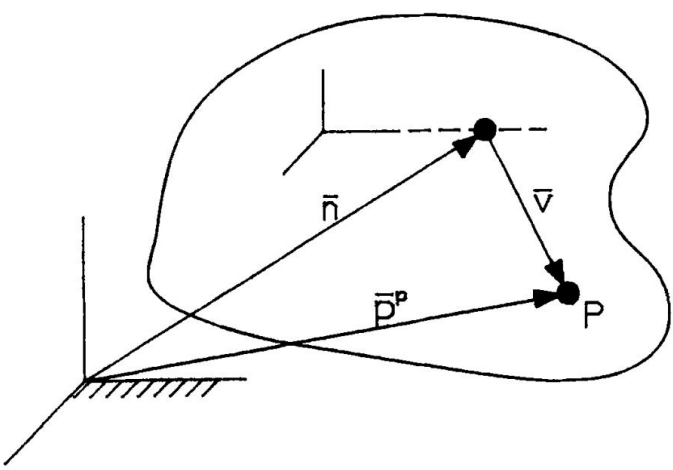

Figure 1. A single flexible body.

computed. The simplest way of writing the variation is:

$$\delta \int_t KEdt = \int_t \int_{vol} \rho^A D\vec{p}^p \cdot {}^A\delta^A D\vec{p}^p d_{vol} dt = -\int_t \int_{vol} \rho^A D^A D\vec{p}^p \cdot {}^A\delta\vec{p}^p d_{vol} dt. \tag{34}$$

In the rigid body case, the quantity ${}^A\delta\vec{p}^p$ is computed from Eq. 24. In continuous elastic systems, there are an infinite number of u. Define the elastic deflection of point p as vector $\vec{v}$, and define points n on a neutral body so that points n and p coincide when $\vec{v} = 0$. If a coordinate system is established in body B and $\vec{p}^p$ is written as the sum of vectors $\vec{n}$ and $\vec{v}$ shown in Figure 1:

$${}^A\delta\vec{p}^p = {}^A\delta\vec{n} + {}^A\delta\vec{v} \tag{35}$$

${}^A\delta\vec{n}$ for all n is a function of a finite number of independent u quantities. The terms ${}^A\delta\vec{v}$ in Eq. 35 are considered arbitrary, subject to boundary conditions.

Although Eq. 34 is compact, it may not be useful. An alternative form of Eq. 34 follows. Note the ease with which the equation is changed. Searching for reductions typically requires expanding, redistributing, and precipitating terms many times, looking for identities and common terms. Therefore the ease with which equations are reduced is related to the ease of their manipulation.

Begin with the position vector defined as in Figure 1. The velocity of a material point is:

$${}^A\vec{V}^p = {}^A\vec{V}^n + {}^A\vec{\omega}^B \times \vec{v} + {}^BD\vec{v}. \tag{36}$$

The kinetic energy is:

$$\begin{aligned} KE = KE_N + \frac{1}{2}\int_{vol} \rho[& {}^A\vec{\omega}^B \times \vec{v} \cdot {}^A\vec{\omega}^B \times \vec{v} + {}^BD\vec{v} \cdot {}^BD\vec{v} \\ & + 2({}^A\vec{V}^n \cdot {}^BD\vec{v} + {}^A\vec{V}^n \cdot {}^A\vec{\omega}^B \times \vec{v} + {}^BD\vec{v} \cdot {}^A\vec{\omega}^B \times \vec{v})]\, d_{vol}. \end{aligned} \tag{37}$$

The term KE_N is the rigid body kinetic energy terms that would exist if the body had no elastic displacement. The variation of kinetic energy can be expressed as:

$$\begin{aligned} \delta KE = \delta KE_N + \int_{vol} \rho[& {}^A\vec{\omega}^B \times \vec{v} \cdot {}^i\delta({}^A\vec{\omega}^B \times \vec{v}) + {}^BD\vec{v} \cdot {}^m\delta^B D\vec{v} \\ & + {}^i\delta^A\vec{V}^n \cdot {}^A\vec{\omega}^B \times \vec{v} + {}^A\vec{V}^n \cdot {}^i\delta({}^A\vec{\omega}^B \times \vec{v}) \\ & + {}^j\delta^A\vec{V}^n \cdot {}^BD\vec{v} + {}^A\vec{V}^n \cdot {}^j\delta^B D\vec{v} \\ & + {}^k\delta^B D\vec{v} \cdot {}^A\vec{\omega}^B \times \vec{v} + {}^BD\vec{v} \cdot {}^k\delta({}^A\vec{\omega}^B \times \vec{v})]\, d_{vol}. \end{aligned} \tag{38}$$

The terms ${}^{i}\delta$, ${}^{j}\delta$, ${}^{k}\delta$, ${}^{l}\delta$, and ${}^{m}\delta$ denote variations in five different coordinate systems to illustrate that terms need not have their variations computed in the same coordinate system. The reason one would introduce a variation in some coordinate system other than an inertial frame is because it may be easier to evaluate variations in rotating coordinates, depending on the specific problem.

Now consider the problem of manipulating Eq. 38 into another form, to demonstrate the ease of manipulation. With the identity from Eq. 6 the variation of kinetic energy is:

$$
\begin{aligned}
\delta KE = \delta KE_N + \int_{vol} \rho[&({}^{A}\vec{\omega}^{B} \times \vec{v} + {}^{A}\vec{V}^{n} + {}^{B}D\vec{v}) \cdot {}^{c}\delta({}^{A}\vec{\omega}^{B} \times \vec{v}) \\
&+ ({}^{B}D\vec{v} + {}^{A}\vec{V}^{n} + {}^{A}\vec{\omega}^{B} \times \vec{v}) \cdot {}^{c}\delta^{B}D\vec{v} \\
&+ ({}^{A}\vec{\omega}^{B} \times \vec{v} + {}^{B}D\vec{v}) \cdot {}^{c}\delta^{A}\vec{V}^{n} \\
&+ ({}^{A}\vec{\omega}^{B} \times \vec{v}) \cdot [{}^{l}\vec{\theta}^{c} \times ({}^{A}\vec{\omega}^{B} \times \vec{v})] \\
&+ {}^{A}\vec{V}^{n} \cdot {}^{i}\vec{\theta}^{c} \times ({}^{A}\vec{\omega}^{B} \times \vec{v}) \\
&+ {}^{B}D\vec{v} \cdot {}^{k}\vec{\theta}^{c} \times ({}^{A}\vec{\omega}^{B} \times \vec{v}) \\
&+ {}^{B}D\vec{v} \cdot {}^{m}\vec{\theta}^{c} \times {}^{B}D\vec{v} + {}^{A}\vec{V}^{n} \cdot {}^{j}\vec{\theta}^{c} \times {}^{B}D\vec{v} \\
&+ {}^{k}\vec{\theta}^{c} \times {}^{B}D\vec{v} \cdot {}^{A}\vec{\omega}^{B} \times \vec{v} \\
&+ {}^{A}\vec{\omega}^{B} \times \vec{v} \cdot {}^{i}\vec{\theta}^{c} \times {}^{A}\vec{V}^{n} + {}^{B}D\vec{v} \cdot {}^{j}\vec{\theta}^{c} \times {}^{A}\vec{V}^{n}]\, d_{vol}.
\end{aligned}
\tag{39}
$$

This is easily reduced to:

$$
\delta KE = \delta KE_N + \int_{vol} \rho[{}^{A}\vec{V}^{p} \cdot {}^{c}\delta({}^{A}\vec{\omega}^{B} \times \vec{v} + {}^{B}D\vec{v} + {}^{A}\vec{V}^{n}) - {}^{A}\vec{V}^{n} \cdot {}^{c}\delta^{A}\vec{V}^{n}]\, d_{vol}. \tag{40}
$$

Using the relations in Eq. 16, Eq. 40 can be written:

$$
\begin{aligned}
\delta KE &= \delta KE_N + \int_{vol} \rho[{}^{A}\vec{V}^{p} \cdot {}^{A}\delta({}^{A}\vec{\omega}^{B} \times \vec{v} + {}^{B}D\vec{v} + {}^{A}\vec{V}^{n}) \\
&\quad + {}^{A}\vec{V}^{p} \cdot {}^{c}\vec{\theta}^{a} \times {}^{A}\vec{V}^{p} \\
&\quad - {}^{A}\vec{V}^{n} \cdot {}^{a}\delta^{A}\vec{V}^{n} - {}^{A}\vec{V}^{n} \cdot {}^{c}\vec{\theta}^{a} \times {}^{A}\vec{V}^{n}]\, d_{vol} \\
&= \delta KE_N + \int_{vol} \rho[{}^{A}\vec{V}^{p} \cdot {}^{A}\delta({}^{A}\vec{V}^{p}) + {}^{c}\vec{\theta}^{a} \cdot {}^{A}\vec{V}^{p} \times {}^{A}\vec{V}^{p} \\
&\quad - {}^{A}\vec{V}^{n} \cdot {}^{a}\delta^{A}\vec{V}^{n} - {}^{c}\vec{\theta}^{a} \cdot {}^{A}\vec{V}^{n} \times {}^{A}\vec{V}^{n}]\, d_{vol} \\
&= \delta KE_N + \int_{vol} \rho[{}^{A}\vec{V}^{p} \cdot {}^{A}\delta({}^{A}\vec{V}^{p}) \\
&\quad - {}^{A}\vec{V}^{n} \cdot {}^{a}\delta^{A}\vec{V}^{n}]\, d_{vol}
\end{aligned}
\tag{41}
$$

$$= \int_{vol} \rho[{}^A\vec{V}^p \cdot {}^A\delta({}^A\vec{V}^p)]\, d_{vol}$$

which is identical to the results in Eq. 34.

CONCLUSIONS

The process demonstrated in this article is an alternative method for accommodating rotating coordinate systems and nonholonomic constraints in energy based formulations of equations of motion. The existing energy methods accommodate rotating coordinates by introducing variable coordinate transformations to express energy as scalar functions in inertial reference systems. When the energy terms are differentiated, the coordinate transforms complicate the formulation. The technique does not require introduction of transformations until the equations have been obtained. This results in equations that assume a simplified form amenable to closed form reduction.

Previous energy techniques applied to nonholonomic systems required the introduction of matrices relating the dependent and independent generalized coordinates. The process shown accommodates nonholonomic constraints by expressing velocity in terms of a subset of the generalized coordinate derivatives. The subset chosen must satisfy identically all nonholonomic constraint equations. The variation of position vectors are related to partial derivatives of the velocities, thereby expressing an arbitrary variation that automatically satisfies all constraints.

By defining generalized speeds as functions of the generalized coordinates, coordinate derivatives, and time, the resulting equations appear in a first order form ideal for computer integration.

The method can be used to derive Kane's equations for rigid systems, not in itself a contribution since the Passerello–Huston equations verified that Kane's method has an energy basis. The demonstrated process, however, does not have the practical difficulties associated with the Passerello–Huston equations for deriving equations using energy principles.

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